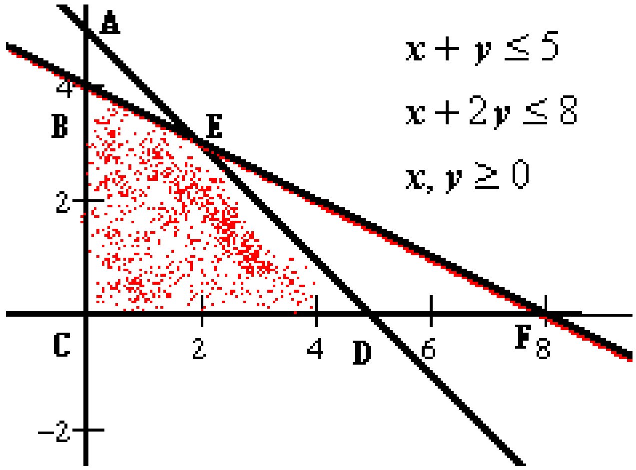
Graphical Solution: 2 variables

Def.: **Feasible region**: set of points that make all linear inequalities in the system true simultaneously.



Def.: Give a feasible region in the plane defined by $\le$ or $\ge$ constraints. A point of intersection of two boundary lines that is also part of the feasible region is called a **corner point (vertex) of the feasible region**. In the above, **B, C, D, E** are vertices for the feasible region shaded in red.

The Fundamental Theorem of Linear Programming: If the feasible region to any linear programming problem has at least one point and is *convex* and if the objective function has a maximum (or minimum) value within the feasible region, then the maximum (or minimum) will always occur at a corner point in that region.

Criteria for the existence of solutions, $f(x,y)=ax+by$

- If the feasible region is bounded, then has a maximum and minimum.
- If the feasible region is unbounded and a, $b>0$, then f has a minimum; but not a maximum.

Click here to get a rough idea why the above holds.

Example

Problem 1
Problem 2
Problem 3

Return to Solving Linear Programming

Why?

Suppose $f(x, y) = Ax + By + C$, a linear function in ***x*** and ***y***, is defined on a line segment. ***f*** obtains its max/min at the end points of the segment.

Ex. $f(x, y) = 2x + 3y$ on the line segment $y = x + 2$ from **(2, 4)** to **(5, 7)**.
Consider the following table of values consisting of various points on the line segment from **(2, 4)** to **(5, 7)** along with the corresponding values of the linear function evaluated on that line segment.

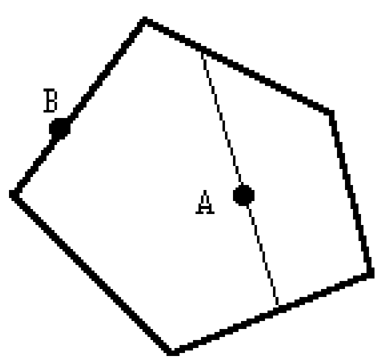
(x,y)	**f(x,y)**
(2,4)	16
(3,5)	21
(4,6)	26
(5,7)	31

So, one can see that as a point moves along the line segment from **(2, 5)** to **(5, 7)**, the values of the objective function increase from **16** (minimum) to **31** (maximum).
In general, given $f(x, y) = Ax + By + C$ defined on a line segment described by $y = mx + b$, then the following can be done:

$$\begin{aligned} f(x, y) &= f(x, mx + b) \\ &= Ax + B(mx + b) + C \\ &= (A + Bm)x + Bb + C \\ F(x) &= A'x + C' \end{aligned}$$

Now, ***F*** is linear in ***x*** and as ***x*** increases from left to right, ***F*** either increases or decreases. Consequently, the maximum and minimum values of ***F*** and, therefore, of ***f***, must occur at the extreme values of ***x***. Hence, the maximum and minimum values of a linear function which is evaluated on a line segment must occur at the endpoints of the line segment.

Suppose ***f*** is defined over a convex polygon.



Is it possible for a maximum/minimum to occur within the polygon, say at **A**? If so, then one can construct a line segment through **A** which intersects on the perimeter. But the

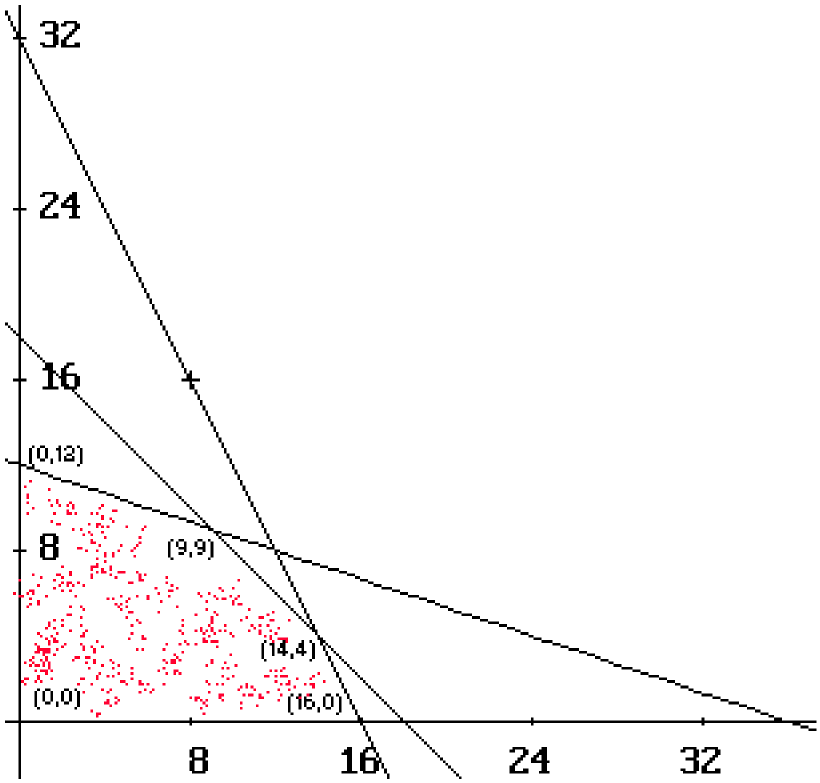
maximum/minimum must occur at the endpoints. Therefore, the maximum/minimum cannot be within the polygon. Is it possible for a maximum/minimun to occur on the perimeter; but not at a vertex (**B**)? No, because a maximum/minimum must occur at the endpoints. Therefore, the maximum/minimum must occur at the vertices.

Return to Graphical Solution

Example

Maximize $f(x, y)=80x+70y$ subject to the constraints: $\begin{cases} 2x+y \le 32 \\ x+y \le 18 \\ x+3y \le 36 \\ x, y \ge 0 \end{cases}$

The corresponding system of linear equations is $\begin{cases} y=-2x+32 \\ y=-x+18 \\ y=-\frac{1}{3}x+12 \end{cases}$ and the polygon which represents the feasible region is



The following table contains the values of the function at the vertices:

(x,y)	f(x,y)
(0,0)	0
(16,0)	1280
(14,4)	1400
(9,9)	1350
(0,12)	840

The maximum value of the function is 1400 and occurs at **(14,4)**.

Return to Graphical Solution

Maximize ***f*(*x*, *y*)=143*x*+60*y*** subject to the constraints:
$$\begin{cases} x + y \le 100 \\ 120x + 210y \le 15000 \\ 110x + 30y \le 4000 \\ x, y \ge 0 \end{cases}$$

Click here to see the solution

Return to Graphical Solution

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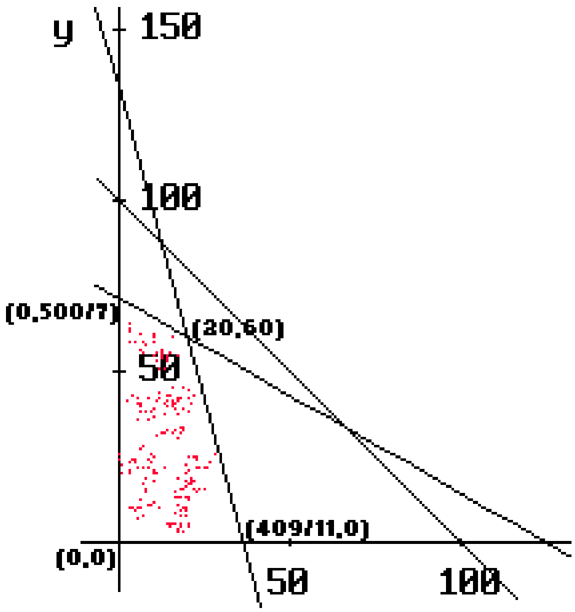
$$\begin{cases} x + y \le 100 \\ 120x + 210y \le 15000 \\ 110x + 30y \le 4000 \\ x, y \ge 0 \end{cases}$$

Consider the corresponding system of linear equations and the feasible region:

$$\begin{cases} x + y = 100 \\ 120x + 210y = 15000 \\ 110x + 30y = 4000 \\ x, y = 0 \end{cases}$$



Click here to see the next step

Return to Graphical Solution

Problem 1 of 3

Maximize $f(x, y) = 143x + 60y$ subject to the constraints:

$$\begin{cases} x + y \le 100 \\ 120x + 210y \le 15000 \\ 110x + 30y \le 4000 \\ x, y \ge 0 \end{cases}$$

Consider the corresponding system of linear equations and the feasible region:

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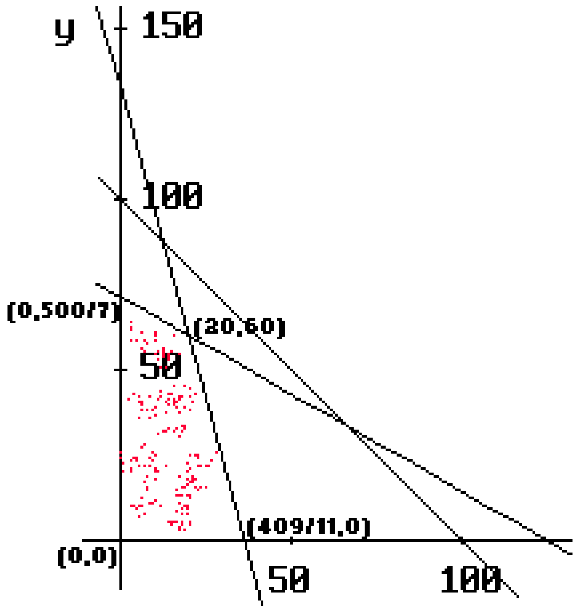

Check the vertices to find that the maximum value is **6460** at **(20,60)**

[Return to Graphical Solution]

Minimize $f(x, y) = 60x + 30y$ subject to the constraints: $\begin{cases} 2x + 3y \geq 120 \\ 2x + y \geq 80 \\ x, y \geq 0 \end{cases}$

Click here to see the solution

Return to Graphical Solution

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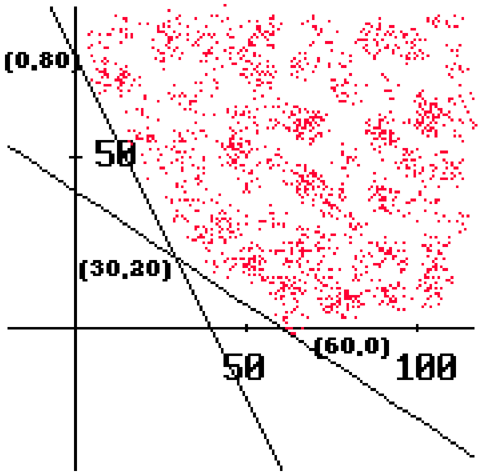


Click here to see the next step

Return to Graphical Solution

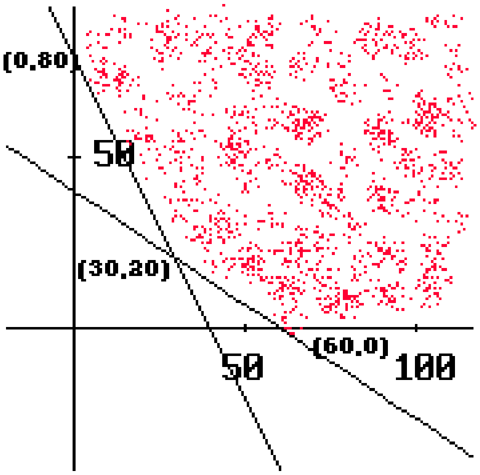
Problem 2 of 3

Minimize ***f*(*x*, *y*)=60*x*+30*y*** subject to the constraints:

$$\begin{cases} 2x+3y \geq 120 \\ 2x+y \geq 80 \\ x, y \geq 0 \end{cases}$$

Consider the corresponding system of linear equations and the feasible region:

$$\begin{cases} 2x+3y = 120 \\ 2x+y = 80 \end{cases}$$



Check the vertices to find that the minimum value is **2400** at **(0,80)** and **(30,20)**.

Return to Graphical Solution

Problem 3 of 3

A pension fund has $30 million to invest. The money is to be divided among Treasury notes, bonds, and stocks. The rules for administration of the fund require that at least $3 million be invested in each type of investment, at least half the money be invested in Treasury notes and bonds, and the amount invested in bonds not exceed twice the amount invested in Treasury notes. The annual yields for the various investments are 7% for Treasury notes, 8% for bonds, and 9% for stocks. How should the money be allocated among the various investments to produce the largets return?

Click here to see the solution

Return to Graphical Solution.

Problem 3 of 3

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In millions of dollars, let ***x*=** the amount in Treasury notes, ***y*=** the amount in bonds,and **30-(*y*+*y*)=** the amount in stocks. The constraints are:

$$\begin{cases} x, y \geq 3 \\ 30-(x+y) \geq 3 \\ x+y \geq 15 \\ y \leq 2x \end{cases}$$

and the objective function to be maximized is:

$$\begin{aligned} f(x, y) &= .07x + .08y + .09[30-(x+y)] \\ &= 2.7 - .02x - .01y \end{aligned}$$

Click here to see the next step.

Return to Graphical Solution

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$$\begin{cases} x, y = 3 \\ y = -x + 27 \\ x + y = 15 \\ y = 2x \end{cases}$$

Click here to see the next step.

Return to Graphical Solution

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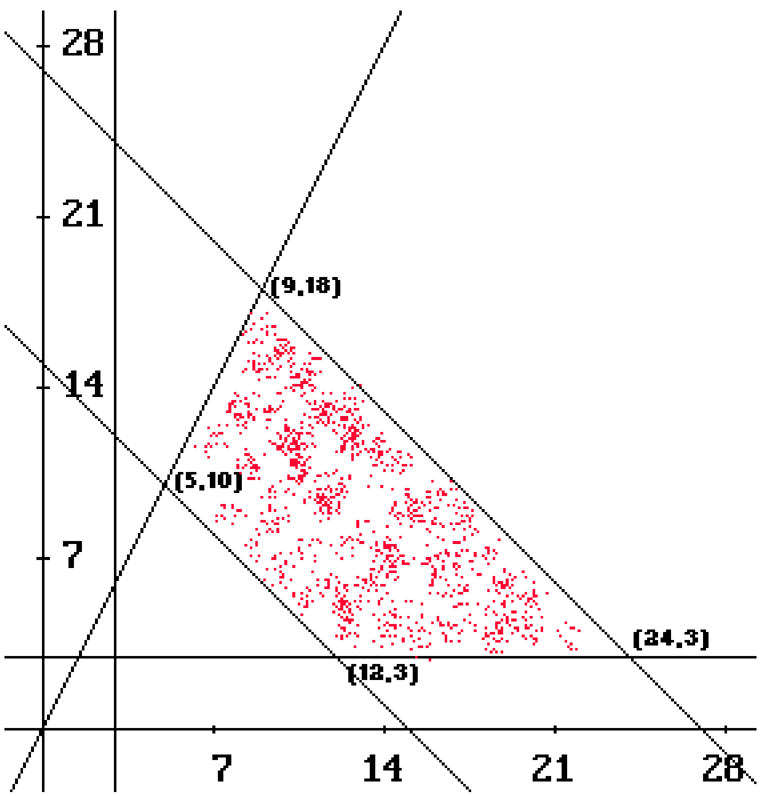

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Return to Graphical Solution

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$$= 2.7 - .02x - .01y$$

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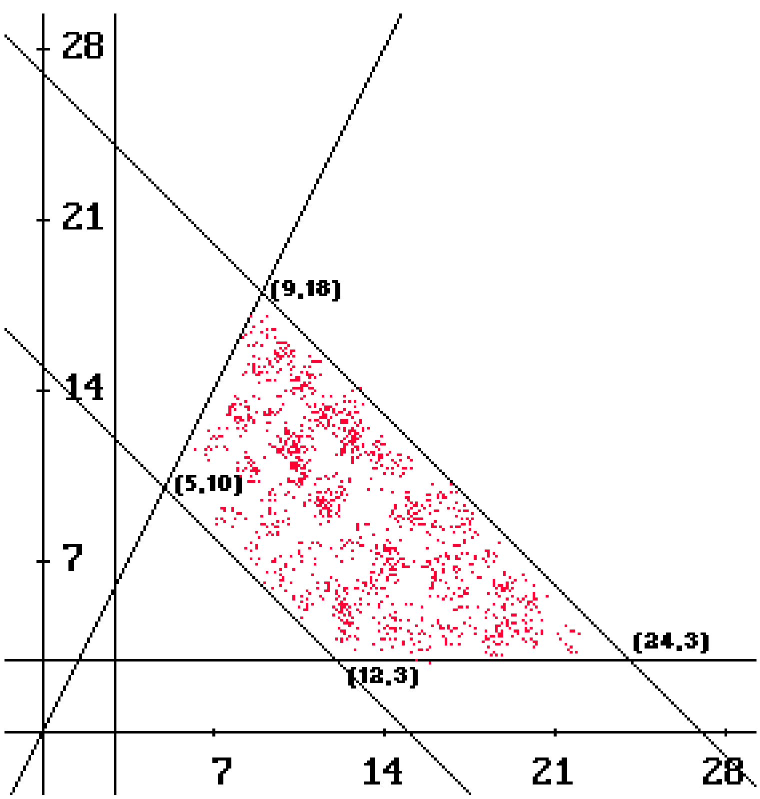

The maximum return is $2.5 million when ***x*=5** million and ***y*=10** million.

Return to Graphical Solution